Clique Is Hard on Average for Unary Sherali-Adams

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EPFL

MIAO Seminar, January 2024

Joint work with Susanna de Rezende and Aaron Potechin

Planted Clique

- Erdős-Rényi random graph $G \sim \mathcal{G}(n, 1/2)$
  - max clique of size $\approx 2 \log n$
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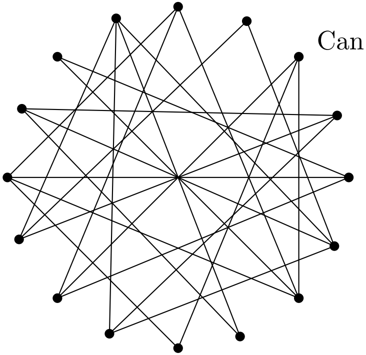
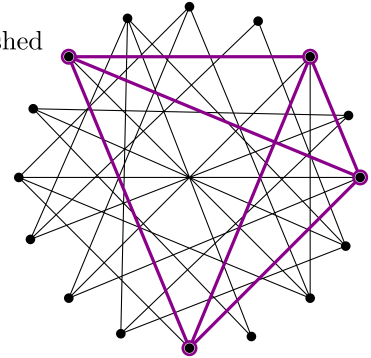
  - $G_0 + K_k$, where $G_0 \sim \mathcal{G}(n, 1/2)$



Can these be distinguished in poly-time?



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- Naïve $n^{O(\log n)}$ algorithm: max clique in $G \sim \mathcal{G}(n, 1/2)$ of size $(2 + o(1)) \log n$
- Poly-time algorithm for $k = \Omega(\sqrt{n})$ [AKS98]
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Theorem (informal)

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Boils down to a size lower bound in unary Sherali-Adams

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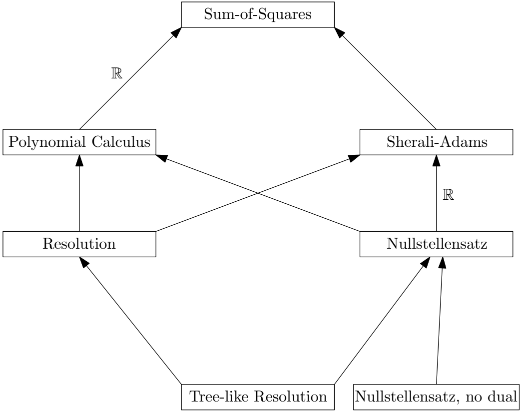
Claim: "$G \sim \mathcal{G}(n, 1/2)$ contains a clique of size $k = n^{1/100}$"



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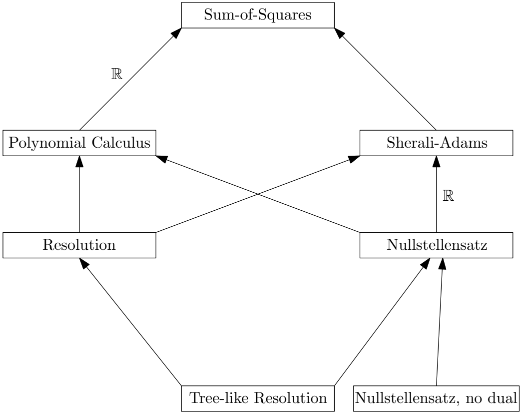
- Optimal under Unique Games Conjecture for many optimization problems
- Captures best algos for clique



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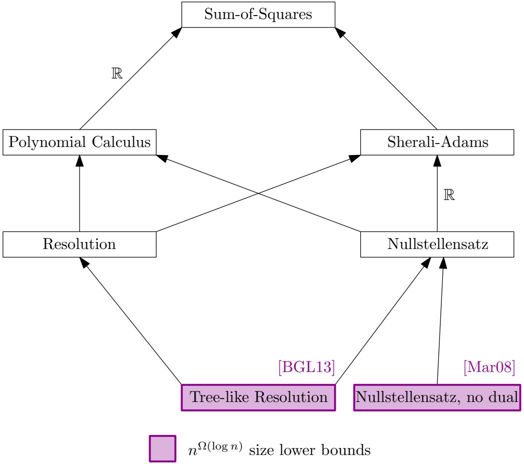

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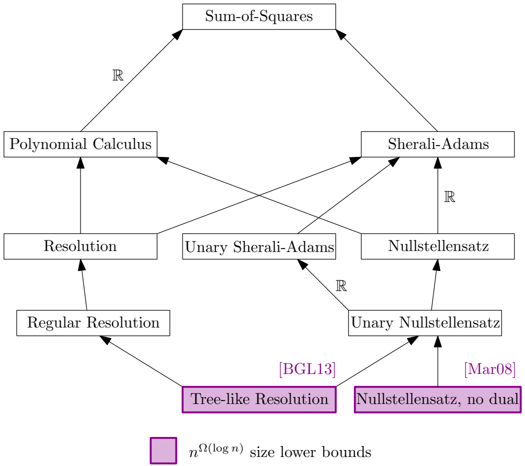

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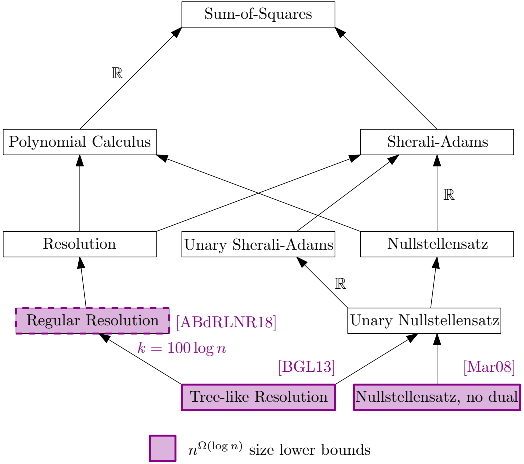

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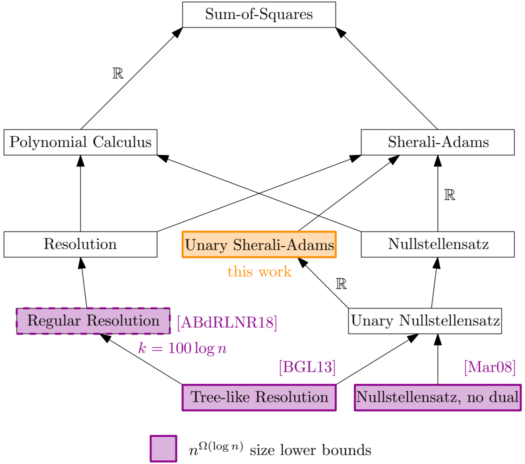

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Results of similar flavor:

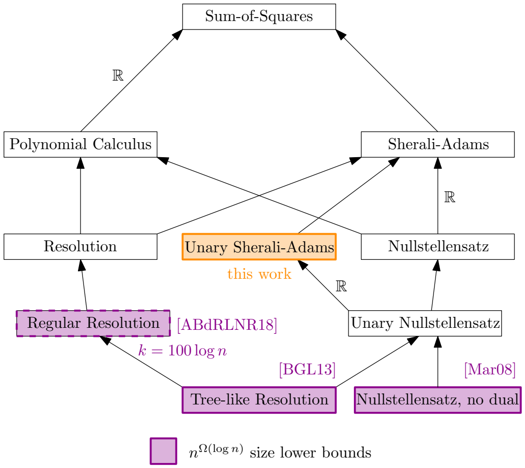
- Monotone & bounded depth circuits

  [Rossman08,Rossman10]

- Resolution:
  - non-tight lower bounds [BIS07,Pang21]
  - weak encoding [LPRT17,DGGM20]

- Degree lower bounds for SoS

  [MPW15,...,BHKKMP19,Pang21]



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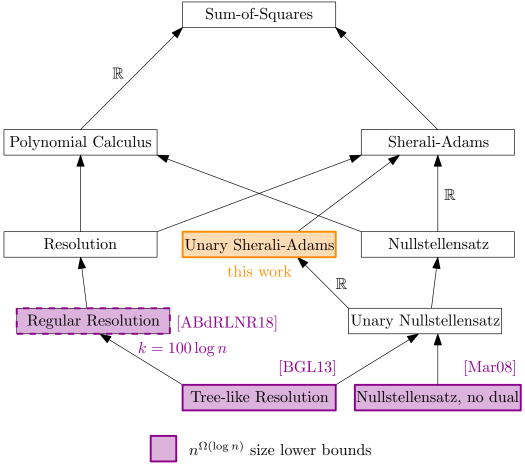

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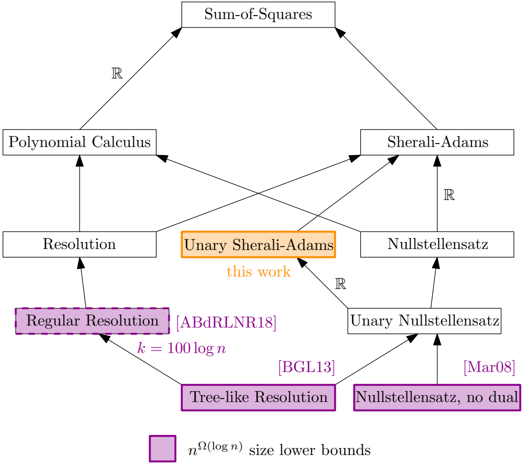

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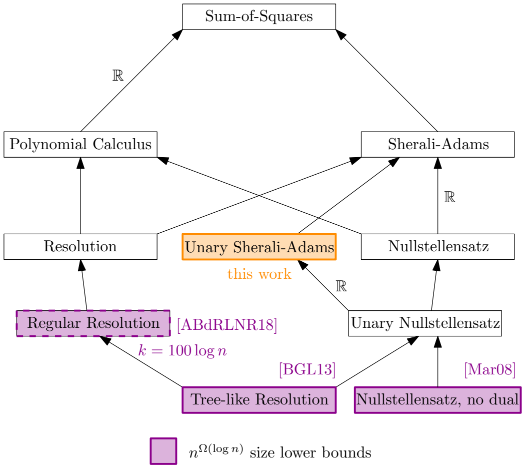

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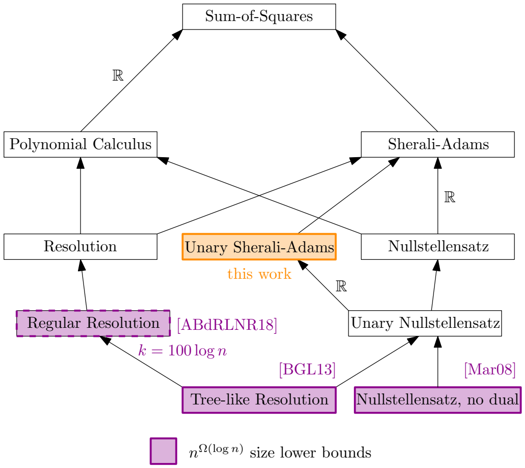

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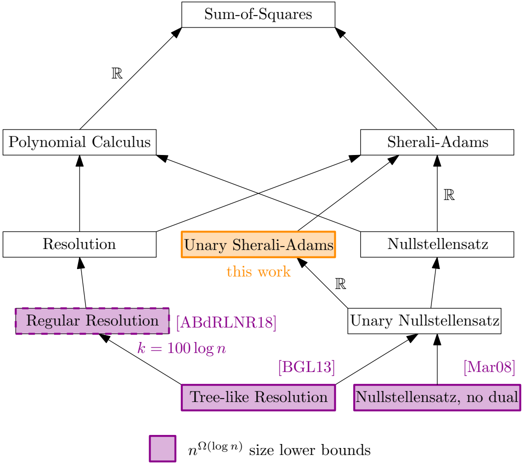
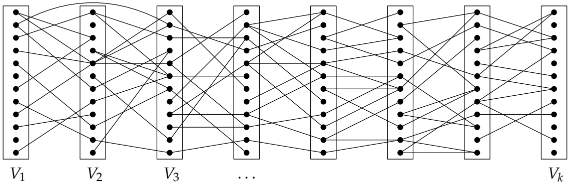
- Seems to require new techniques...



Clique Formula & unary Sherali-Adams

Clique Formula: Block Encoding

Encode claim k-partite graph G contains a k-clique as the polynomial system clique(G, k)



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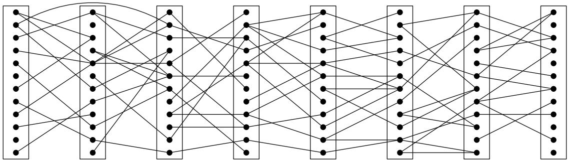

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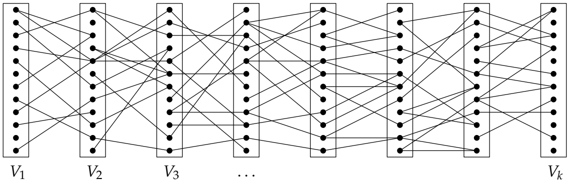

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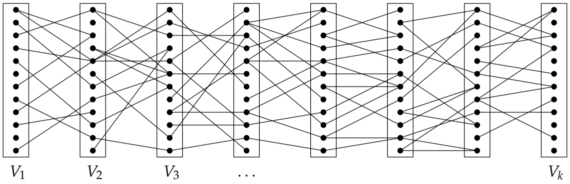

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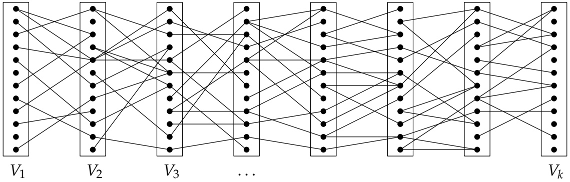

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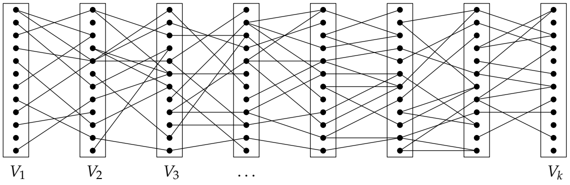

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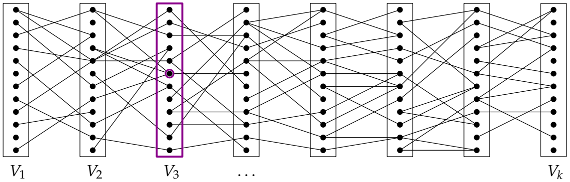

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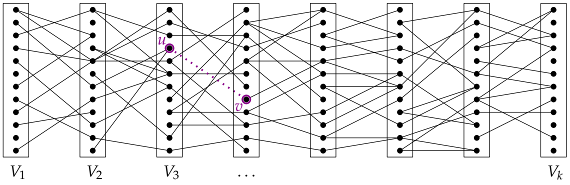

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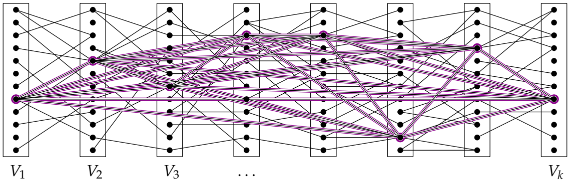

clique(G, k) sat if and only if there is a k-clique with a single vertex per block

The Unary Sherali-Adams Proof System

- Boolean variables $x_1, \ldots, x_m, \bar{x}_1, \ldots, \bar{x}_m$
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$$\sum_{i \in [m]} q_i \, p_i + \sum_{\substack{A,B \subseteq [n] \\ c_{A,B} \geq 0}} c_{A,B} \prod_{i \in A} x_i \prod_{j \in B} \bar{x}_j = -M$$

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- The size of such a refutation is the sum of the magnitude of all coefficients

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- Let $\mathcal{G}(n, k, p)$ be distribution over k-partite graphs, partitions of size n, include edge $e = \{u, v\}$ with probability p iff u, v in distinct parts

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Today: only $p = 1/2$ and hence $D \approx 2 \log n$

Proof Ideas

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- Implies a $\mu(1)/\delta$ unary Sherali-Adams size lower bound to refute $\mathcal{P}$:

$$\sum_{p_i \in \mathcal{P}} \mu(q_i p_i) + \sum_{\substack{A,B \subseteq [n] \\ c_{A,B} \geq 0}} c_{A,B}\, \mu\Big(\prod_{i \in A} x_i \prod_{j \in B} \bar{x}_j\Big) = -\mu(M)$$

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$$\sum_{p_i \in \mathcal{P}} \sum_{m \in q_i} \underbrace{c_m \, \mu(m \cdot p_i)}_{\geq -|c_m|\delta} + \sum_{\substack{A,B \subseteq [n] \\ c_{A,B} \geq 0}} c_{A,B} \, \mu\Big(\prod_{i \in A} x_i \prod_{j \in B} \bar{x}_j \Big) = -\mu(M)$$

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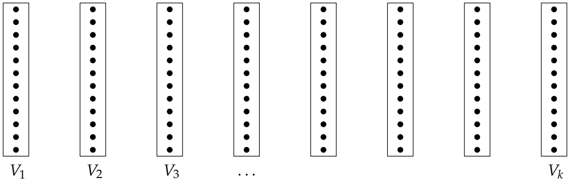

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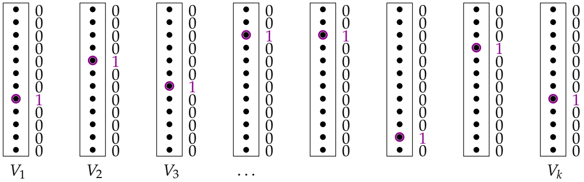

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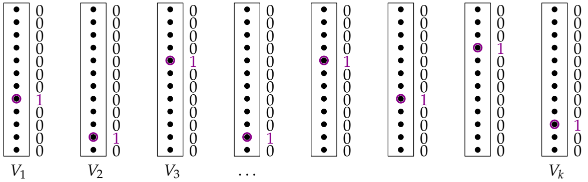

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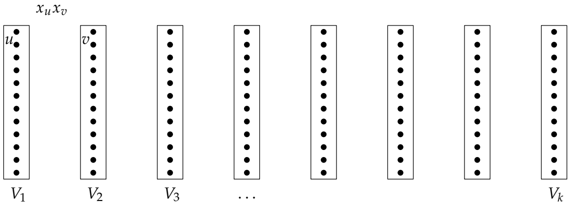

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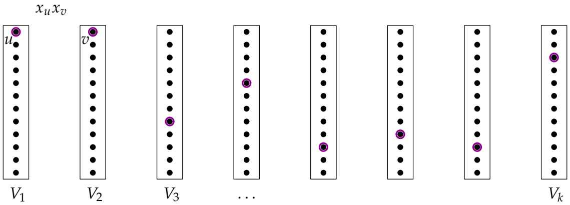

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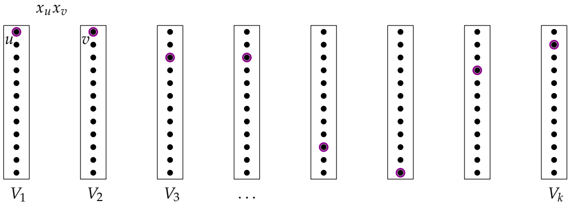

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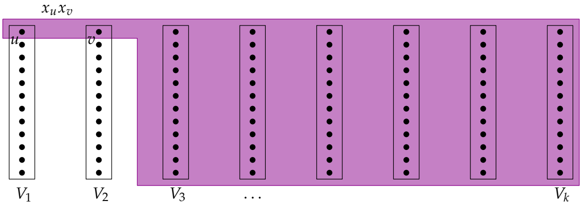

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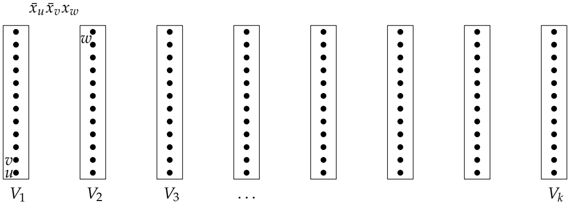

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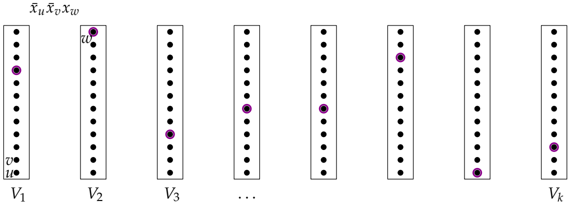

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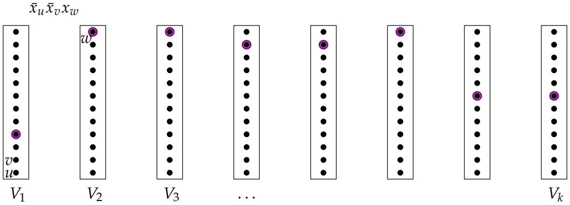

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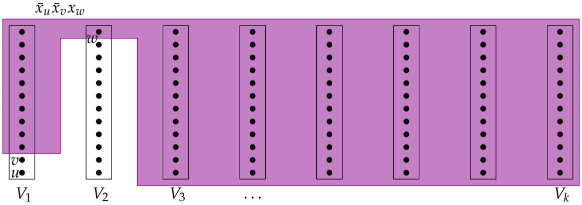

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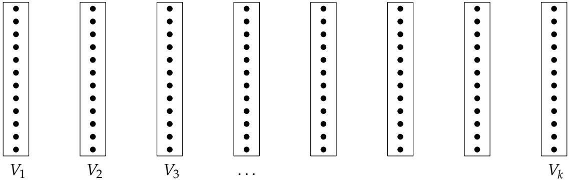

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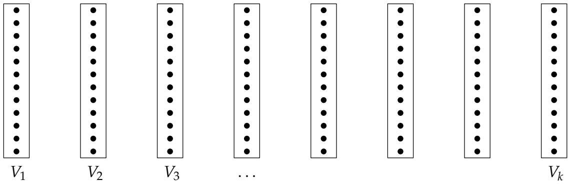

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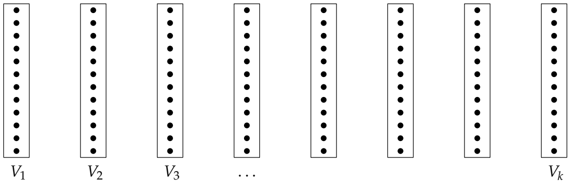

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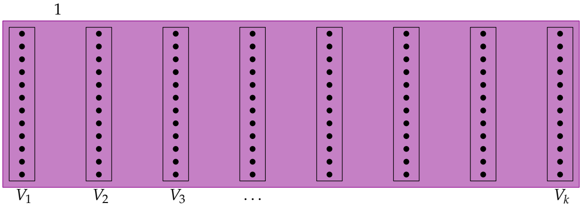

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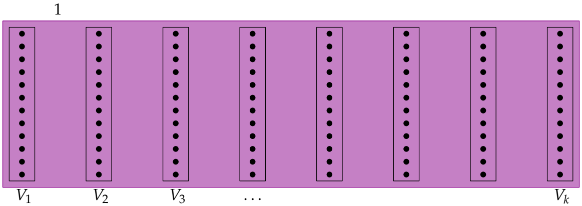

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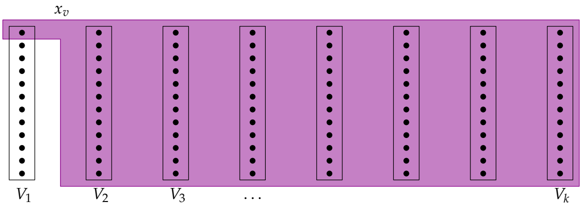

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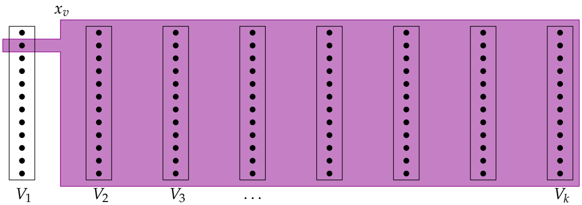

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  - For tuple t relevant assignment ρ_t is $\rho_t(x_v) = 1$ if $v \in t$ and 0 otherwise
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- Attempt 1: $\mu(m) = \frac{|Q(m)|}{n^k}$

- $\mu(1) = 1$ & $\mu(m) \geq 0$
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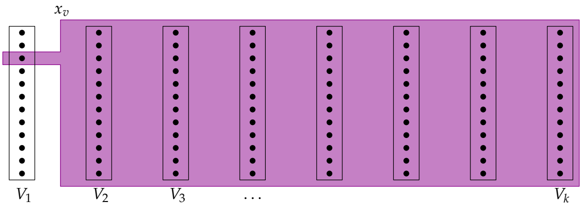

Pseudo-Measure: Construction, Failed Attempt I

Goal

Construct a $n^{-\Omega(\log n)}$-pseudo-measure for clique(G, k), where $G \sim \mathcal{G}(n, k, 1/2)$ and $k \leq n^{0.1}$

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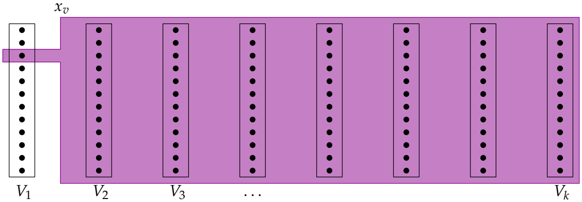

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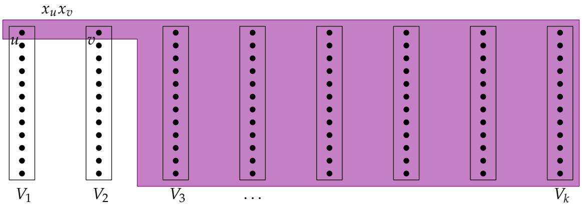

Pseudo-Measure: Construction, Failed Attempt II

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$$\mu_0(m) = n^{-k} \sum_{t \in Q(m)} 2^{\binom{k}{2}} \mathbb{1}_{\{\text{t is clique}\}}(G)$$

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Problem: no k-cliques in the graph!

Pseudo-Measure: Construction, Successful Attempt

Goal

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Interlude: Fourier Characters

Fourier Characters

- Character χ_e for each potential edge $e = \{u, v\}$, i.e., if u, v in distinct blocks,

$$\chi_e(G) = \begin{cases} 1 & \text{if } e \in E(G), \text{ and} \\ -1 & \text{if } e \notin E(G). \end{cases}$$

- For set E of potential edges we let $\chi_E(G) = \prod_{e \in E} \chi_e(G)$. In particular $\chi_\emptyset(G) = 1$.

$$\mu_0(m) = n^{-k} \sum_{t \in Q(m)} 2^{\binom{k}{2}} \mathbb{1}_{\{\text{t is clique}\}}(G)$$

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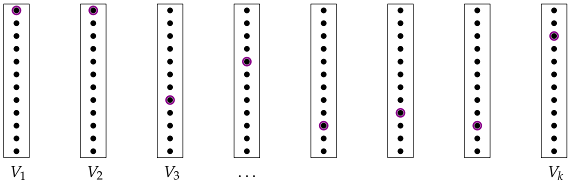

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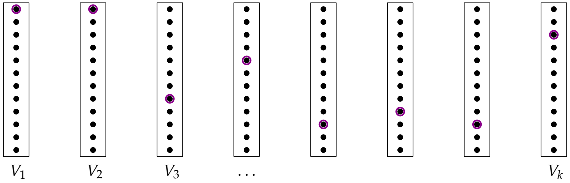

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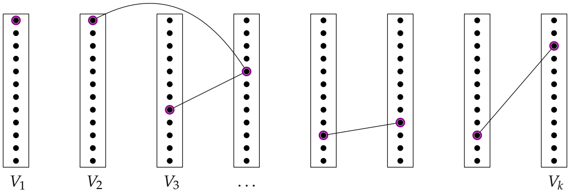

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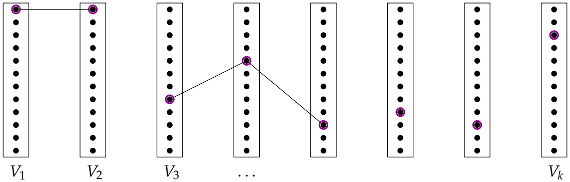

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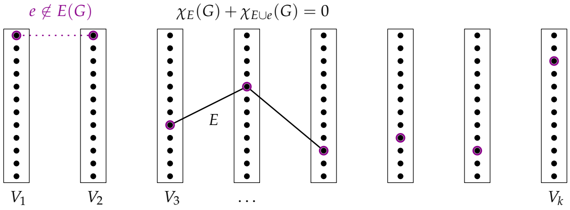

Fourier Characters: Pattern Graphs

Convenient to identify edge sets that "look the same"

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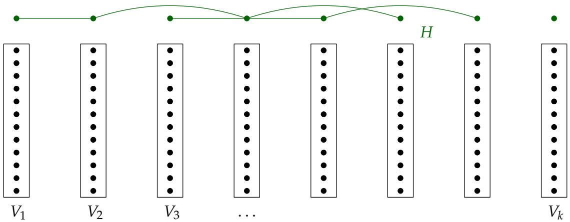

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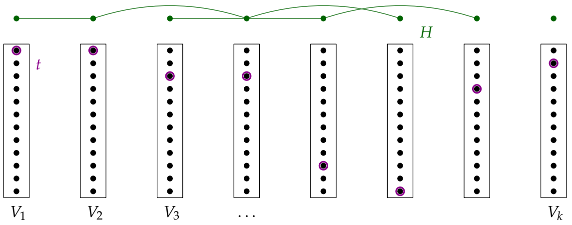

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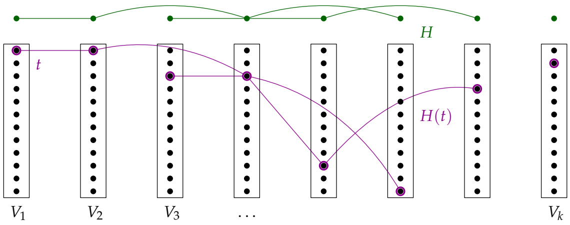

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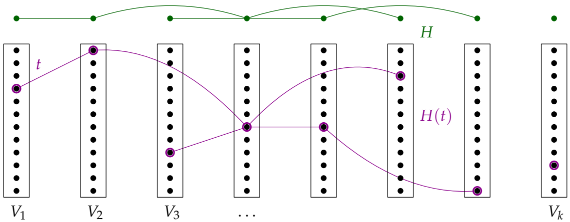

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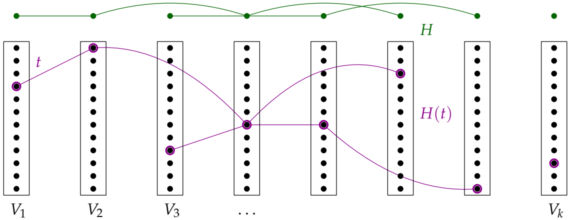

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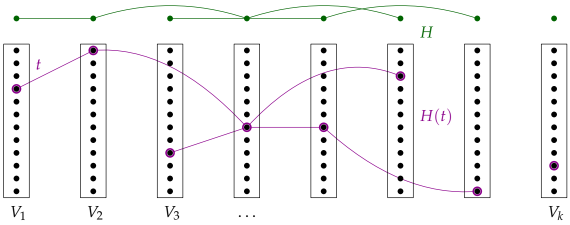

Back to Pseudo-Calibration

Pseudo-Measure by Pseudo-Calibration

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Pseudo-Calibration: 2^{nd} Moment Calculation

Let us analyze the 2^{nd} moment of $\mu_0(1)$; recall that $\mathbb{E}_G[\mu_0(1)] = 1$

$$\mathbb{E}[\mu_0^2(1)] = n^{-2k} \sum_{H,H' \subseteq \binom{k}{2}} \sum_{t,t'} \mathbb{E}[\chi_{H(t)}(G)\chi_{H'(t')}(G)]$$

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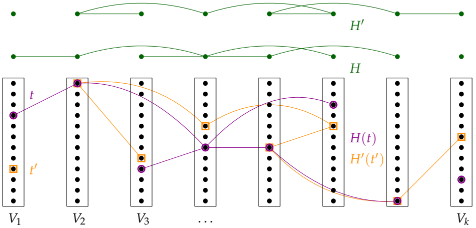

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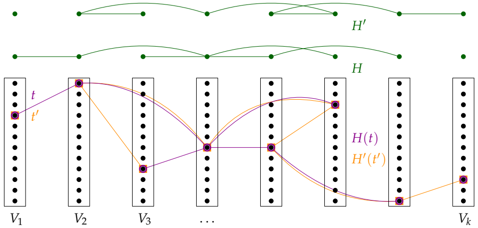

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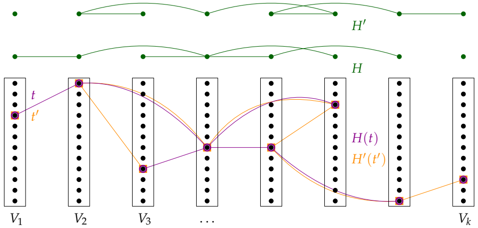

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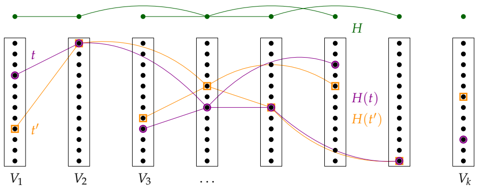

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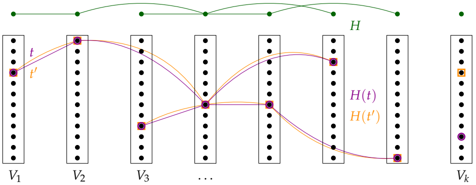

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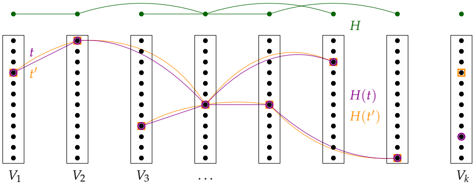

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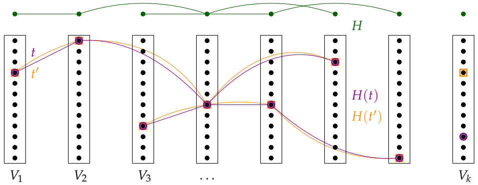

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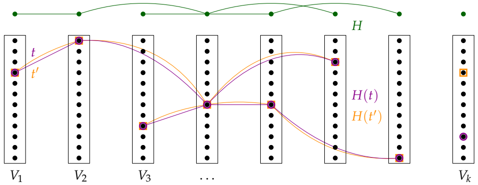

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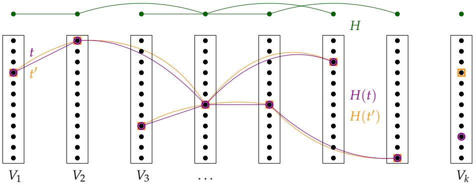

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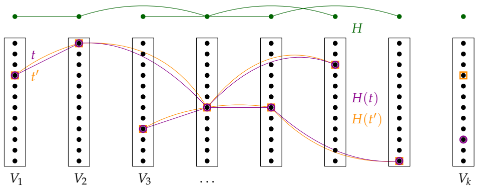

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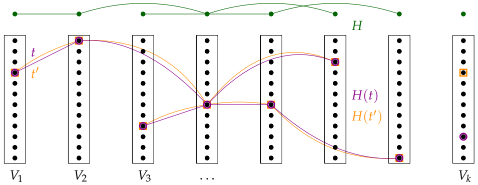

$= 1 + n^{-\Omega(1)}$, if only sum H with $|V(E(H))| \leq \eta \log n$.

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Edge Axioms

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- m monomial; $e = \{v_1, v_2\} \notin E(G)$ for $v_1 \in V_1$ and $v_2 \in V_2$; edge axiom $x_{v_1} x_{v_2}$
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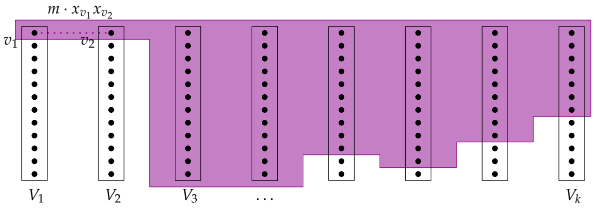

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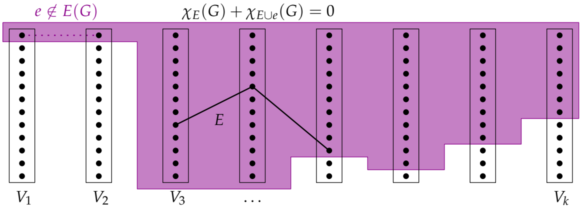

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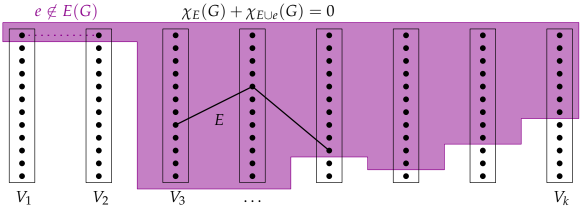

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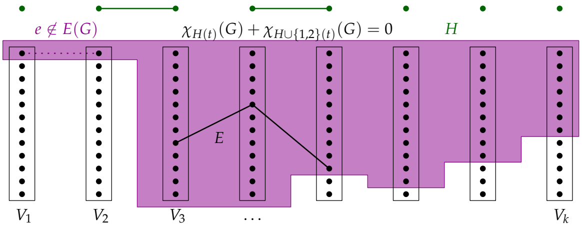

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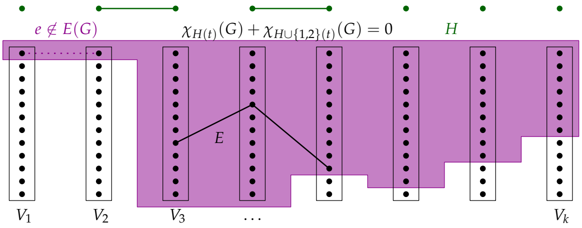

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With high probability over $G \sim \mathcal{G}(n, k, 1/2)$ it holds for any H and Q that

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$$\leq \sum_{\substack{H: \\ \mathrm{vc}(H)=d \\ \{1,2\}\notin H \\ \mathrm{vc}(H\cup\{1,2\})=d+1}} n^{-d/8} \approx 2^{dk} n^{-d/8} \approx n^{\Omega(k)}$$

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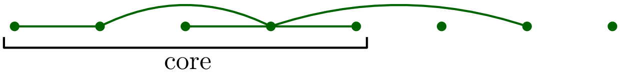

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Back to Edge Axioms

Edge Axioms, Successful Attempt

$$|\mu(m \cdot x_{v_1} x_{v_2})| = n^{-k} \Bigg| \sum_{\substack{H: \\ \mathrm{vc}(H)=d \\ \{1,2\} \notin H \\ \mathrm{vc}(H \cup \{1,2\}) = d+1}} \sum_{t \in Q} \chi_{H(t)}(G) \Bigg|$$

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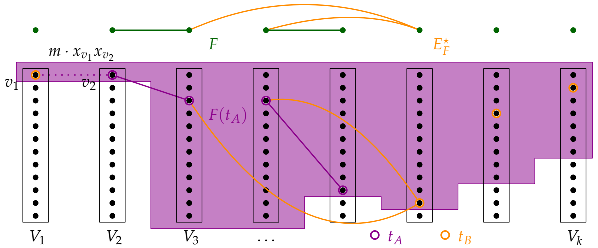

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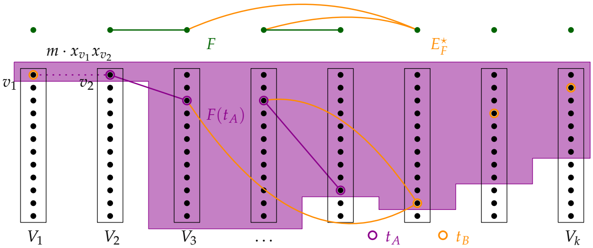

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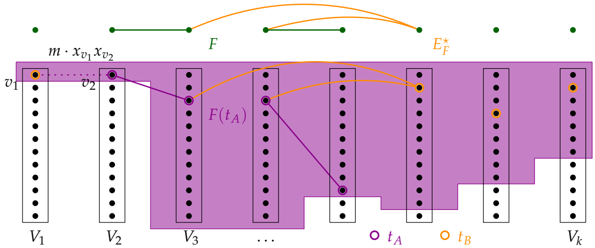

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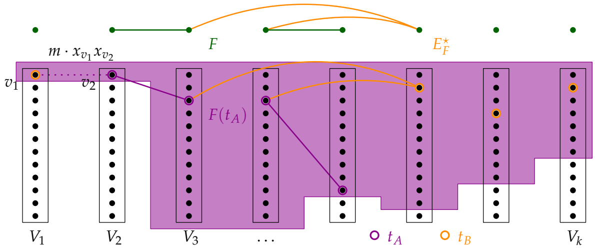

- Fact: common neighborhoods behave as expected in random graphs: for small tuple t, that is, $|t| \le d$, we have

$$|N^\cap(t) \cap V_i| = |\bigcap_{u \in t} N(u) \cap V_i| = (1 \pm n^{-1/5}) \left(\frac{1}{2}\right)^{|t|} n$$

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Summary & Recap

Proof Summary

- Duality gives the notion of a δ-pseudo-measure
- We construct a $n^{-\Omega(\log n)}$-pseudo-measure for clique by Pseudo-Calibration:

$$\mu(m) = n^{-k} \sum_{\substack{H \subseteq \binom{k}{2} \\ \mathrm{vc}(H) \leq d}} \sum_{t \in Q(m)} \chi_{H(t)}(G)$$

- We argued that
  - μ is large on 1: $\mu(1) \approx 1$
  - μ is small on edge-axioms: $|\mu(m \cdot x_u x_v)| \leq n^{-\Omega(\log n)}$
- It remains to argue that
  - μ is basically non-negative: $\mu(m) \geq -n^{-\Omega(\log n)}$

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$$\mathcal{G}(n, 1/2) \approx \mathcal{G}(n, 1/2, n^{1/100})$$

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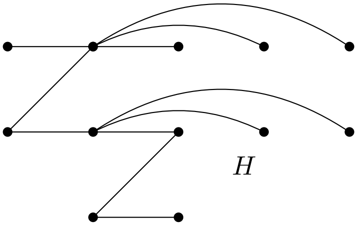
Thanks!

Further Material

Cores

Cores, Construction



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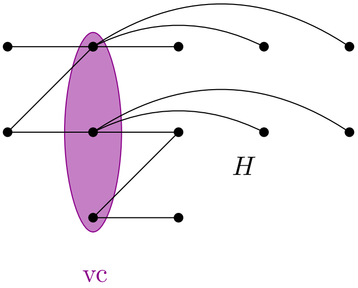

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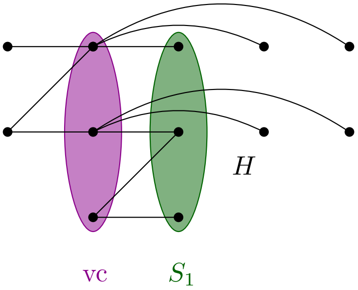

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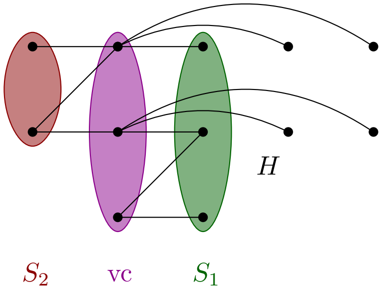

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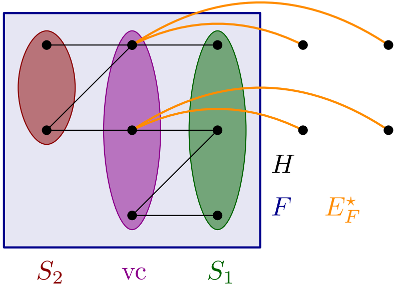

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On the (Almost) Non-Negativity of μ

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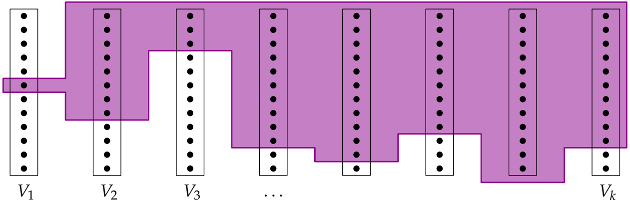

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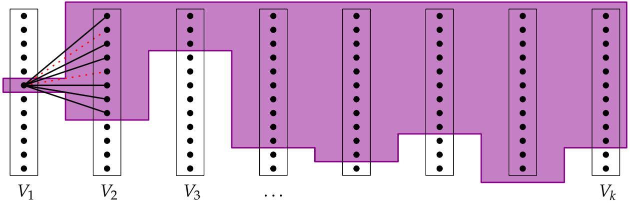

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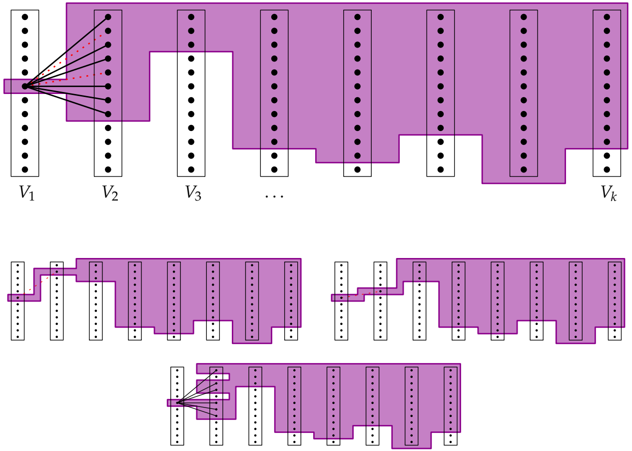

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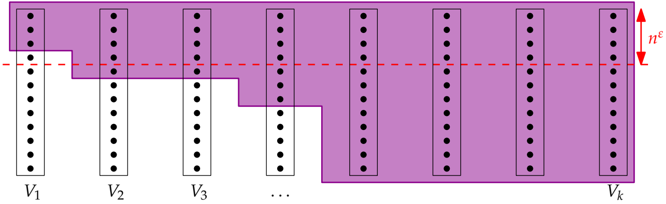

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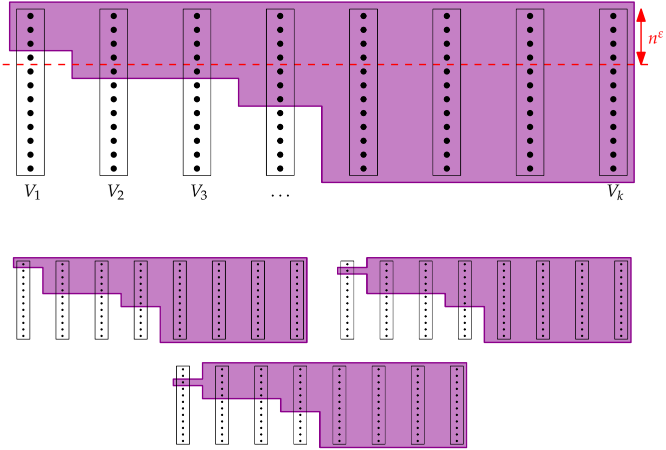

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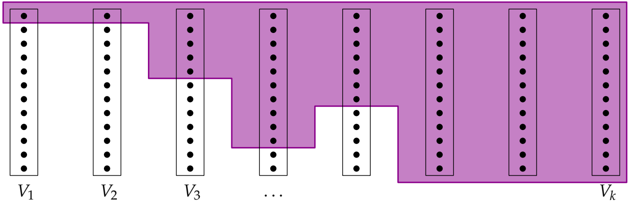

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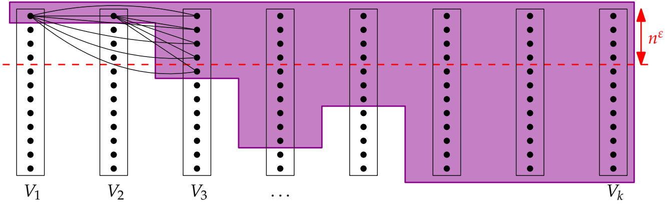

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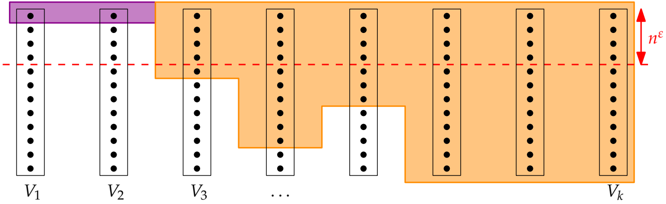

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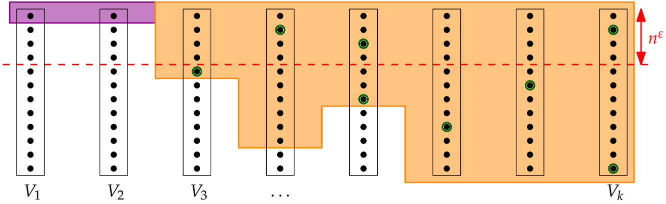

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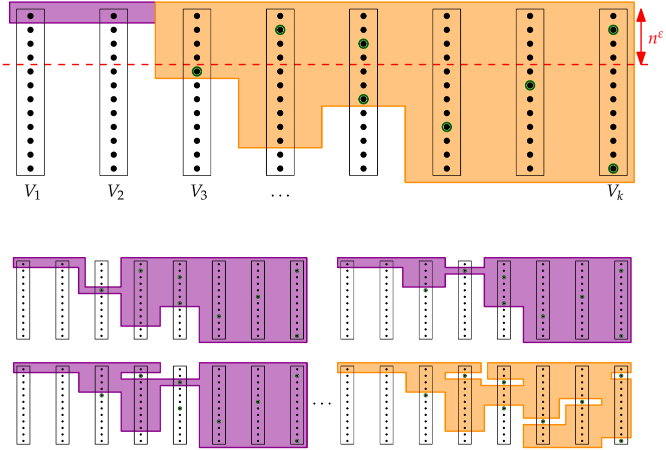

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  - Q is an edge-axiom hence $\mu(Q) \geq -n^{-10\varepsilon \log n}$, or
  - Q is small; $|Q| \approx n^{k-d}$ thus $\mu(Q) \geq -n^{-10\varepsilon \log n}$, or
  - Q has large, well-behaved blocks & singletons adjacent to Q

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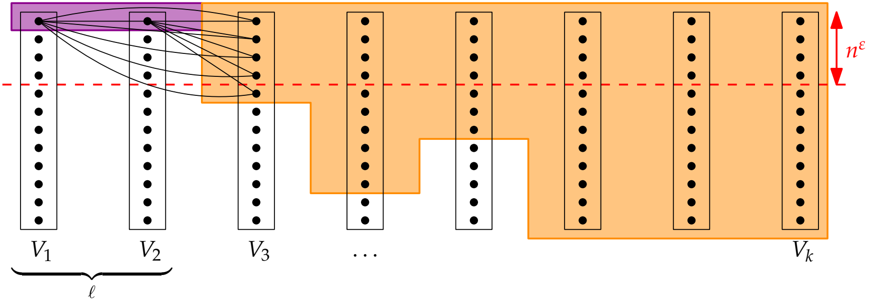
- May conclude for any monomial m that $\mu(m) \geq -n^{-\Omega(\log n)}$

Non-Negativity: Concentration of Measure

Lemma

For any well-behaved rectangle Q with ℓ singletons,

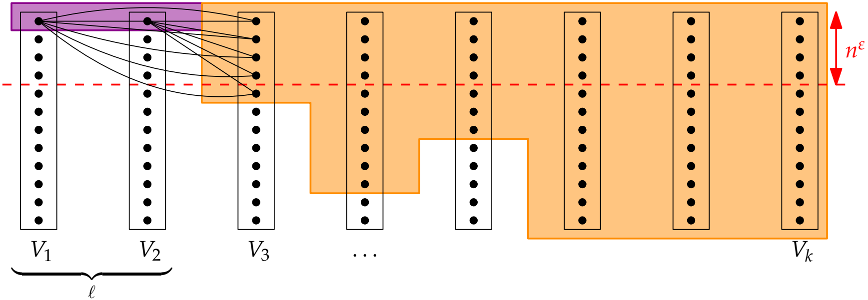


Non-Negativity: Concentration of Measure

Lemma

For any well-behaved rectangle Q with ℓ singletons, with high probability, it holds that

$$\mu(Q) = 2^{\ell(k-(\ell+1)/2)} \cdot |Q|n^{-k} \cdot (1 \pm n^{-\varepsilon})$$



Non-Negativity: Concentration of Measure

Lemma

For any well-behaved rectangle Q with ℓ singletons, with high probability, it holds that

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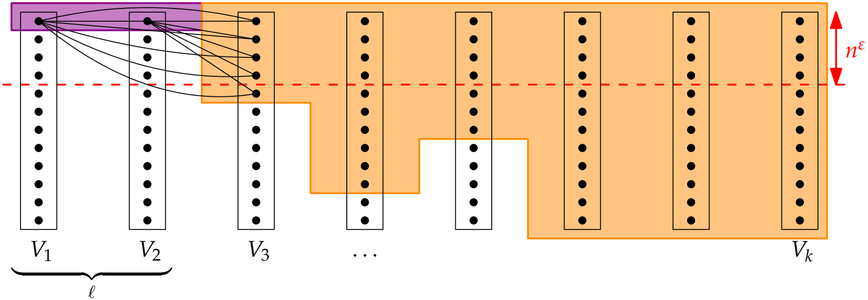
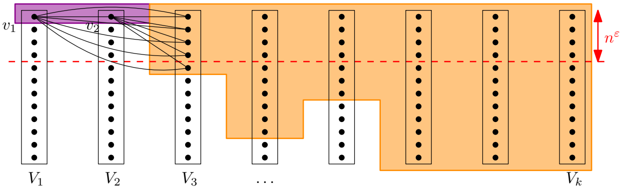
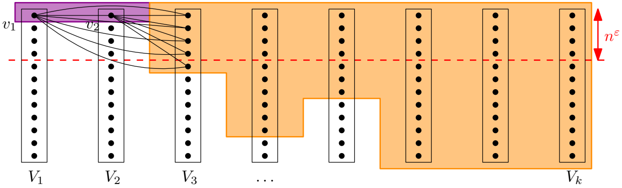


Non-Negativity: Concentration of Measure, Proof Idea



$$\mu(Q) = n^{-k} \sum_{\substack{H: \\ \mathrm{vc}(H) \leq d}} \sum_{t \in Q} \chi_{H(t)}(G)$$

Non-Negativity: Concentration of Measure, Proof Idea



$$\mu(Q) = n^{-k} \sum_{\substack{H: \\ \mathrm{vc}(H) \leq d \\ \{1,2\} \notin H}} \sum_{t \in Q} \chi_{H(t)}(G) + \sum_{\substack{H: \\ \mathrm{vc}(H) \leq d \\ \{1,2\} \in H}} \sum_{t \in Q} \chi_{H(t)}(G)$$

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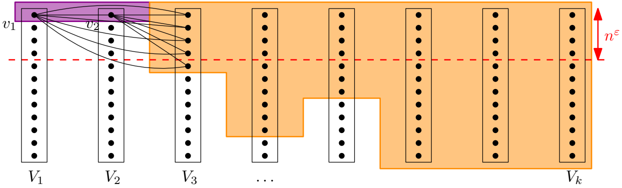

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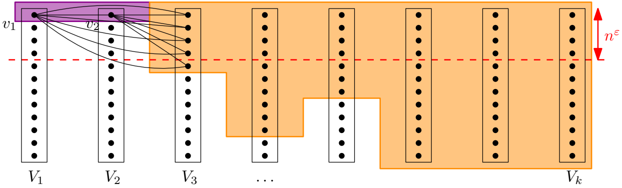

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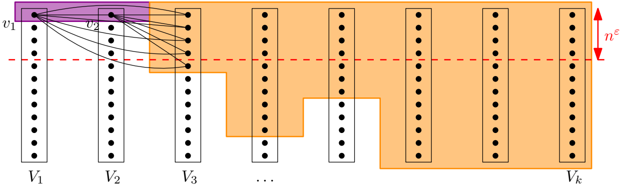
Non-Negativity: Concentration of Measure, Proof Idea



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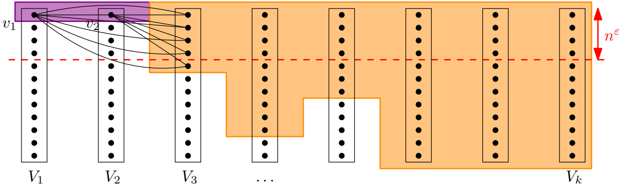

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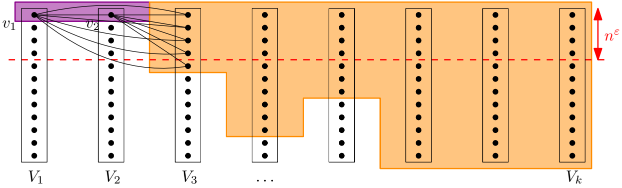

$$\mu(Q) \approx \color{blue}{2} \cdot n^{-k} \sum_{\substack{H: \\ \mathrm{vc}(H) \le d \\ \color{blue}{\{1,2\} \in H}}} \sum_{t \in Q} \chi_{H(t)}(G)$$

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- As $\ell < d$, there is at least one unconditioned edge left

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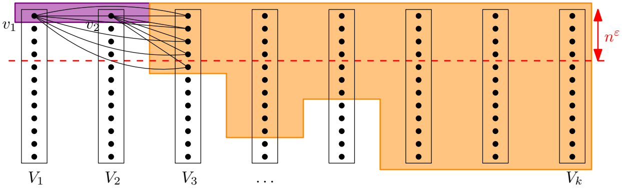

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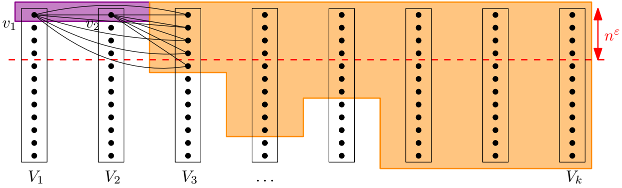

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- Cores with single edge have concentration $(1 \pm n^{-\varepsilon})$